Epsilon Modal Logics

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In this talk, I introduce Epsilon Modal logics, a new class of modal logics I develop which are structurally analogous to Hilbert's Epsilon Calculus:

- In Epsilon Calculus, epsilon terms are introduced which pick a witness for their bound formula, if any;
- In Epsilon Modal logics, 'epsilon' modalities are introduced which pick a related world satisfying their formula index, if any.

EMLs share many properties of Epsilon Calculus to the propositional level. In particular, EMLs are conservative over their Modal logic bases in the same way in which Epsilon Calculus is conservative over Predicate logic.

More interestingly, it turns out that the two systems are embeddable into each other by extending the standard translation and Fitting's modal translation over epsilon terms and modalities resp.

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This correspondence between EMLs and Epsilon Calculus spreads to applications. Epsilon terms have been interpreted as:

- ‘indefinite descriptions’ of objects, in philosophy of language;
- ‘ideal objects’ of mathematical properties in Hilbert’s Program;
- explicit definitions of ‘theoretical terms’ in scientific theories by Carnap.

On intensional grounds, epsilon modalities inherit and generalize these interpretations to:

- ‘indefinite descriptions’ of points of evaluation;
- ‘ideal worlds’ of mathematical structures, as in Concept Structuralism;
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Hilbert and Bernays (1939) developed Epsilon Calculus in the context of the foundational project known as Hilbert's Program (Zach, 2020).

In Epsilon Calculus, a special term-forming operator 'epsilon' is introduced which binds a formula. The interpretation of the resulting 'epsilon terms' is partially open: they represent a witness, if any, of the bound formula.

No further specification on the actual referent of the term is given. This allows epsilon terms to encode quantification. Formally, epsilon terms are usually interpreted over arbitrary choice functions.

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Semantics

Epsilon Calculus extends a language for predicate logic with terms $\varepsilon x\, A$ and $\tau x\, A$, consisting of binders ε and τ resp. binding a variable x in a (open) formula A.

The interpretation of ε- and τ-terms relies on choice functions:

- for any f.o. model $\mathcal{M} = \langle \mathcal{D}, I \rangle$, consider all total choice functions ϕ s.t. $\phi(\emptyset)$ picks an arbitrary object in the domain:

$$\phi(X) := \begin{cases} d \in X & \textit{if } X \neq \emptyset \\ d \in \mathcal{D} & \textit{otherwise} \end{cases}$$

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The interpretation of ε- and τ-terms is thus defined:

- A term $\varepsilon x\, A$ denotes an object satisfying A, if any:

$$I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, A) := \phi\{d \in \mathcal{D} \mid \mathcal{M}, \sigma\tfrac{d}{x}, \phi \Vdash A\}$$

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By totality of ϕ, ε- and τ-terms always denote, and for any $\mathcal{M}$, σ, ϕ:

$$I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, \neg A) = I_{\mathcal{M},\sigma,\phi}(\tau x\, A) \qquad I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, A) = I_{\mathcal{M},\sigma,\phi}(\tau x\, \neg A)$$

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Truth in a model and validity for Epsilon Calculus $\boldsymbol{\varepsilon}\mathbf{P}$ are defined over all choice functions:

$$\mathcal{M} \vDash A \quad \textit{iff} \quad \forall\sigma\forall\phi\colon \mathcal{M}, \sigma, \phi \Vdash A$$

$$\Gamma \vDash_{\boldsymbol{\varepsilon}\mathbf{P}} C \quad \textit{iff} \quad \forall\mathcal{M}\forall\sigma\forall\phi\colon \forall A \in \Gamma\colon \mathcal{M}, \sigma, \phi \Vdash A \Rightarrow \mathcal{M}, \sigma, \phi \Vdash C$$

Referents of ε- and τ-terms hence remain indeterminate in evaluations:

Example

Let $\mathcal{M}$ be an $\boldsymbol{\varepsilon}\mathbf{P}$ model s.t. $\mathcal{D} = \{d_1, d_2, d_3\}$, and $I_{\mathcal{M},\sigma,\phi}(P) = I_{\mathcal{M},\sigma,\phi}(Q) = \{d_1, d_2\}$.

Then, $\mathcal{M} \vDash Q\varepsilon x\, Px$, but the denotation of $\varepsilon x\, Px$ is indeterminate:

- $\phi_1\{d \in \mathcal{D} \mid \mathcal{M}, \sigma\frac{d}{x}, \phi_1 \Vdash Px\} = d_1$ for some ϕ_1;
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Axiomatization

The axiomatization of $\varepsilon\mathbf{P}$ is obtained by adding the following axioms over f.o. (quantifier-free) Predicate logic:

Crit $A(t) \to A(\varepsilon x\, A(x))$

Def $C(\varepsilon x\, \neg A) \leftrightarrow C(\tau x\, A)$

Ext $(A \leftrightarrow B)(\tau x\, (A \leftrightarrow B)/x) \to (C(\varepsilon x\, A) \leftrightarrow C(\varepsilon x\, B))$

No rule of generalization/eigenvariable conditions needed!

Theorem (Soundness and Completeness of $\varepsilon\mathbf{P}$)

$\vDash_{\varepsilon\mathbf{P}} A$ *iff* $\vdash_{\varepsilon\mathbf{P}} A$

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Crit and Def allow defining $\exists$ and $\forall$ quantifiers over epsilon terms:

$$\exists x\, A :\leftrightarrow A(\varepsilon x\, A/x) \qquad \forall x\, A :\leftrightarrow A(\tau x\, A/x)$$

- The referent of $\varepsilon x\, A$ satisfies A in x iff its extension is non-empty, since ϕ would pick a witness for it in this case;
- The referent of $\tau x\, A$ satisfies A in x iff its extension is the domain, since ϕ would pick an arbitrary object in this case.

Ext ensures the extensionality of ϕ. Without it, different witnesses may be chosen over different syntactic form of equivalent bound formulas:

$$I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, A) \neq I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, (A \vee A))$$
$$I_{\mathcal{M},\sigma,\phi}(\tau x\, (A \wedge B)) \neq I_{\mathcal{M},\sigma,\phi}(\tau x\, (B \wedge A))$$
$$I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, ((A \wedge B) \wedge C)) \neq I_{\mathcal{M},\sigma,\phi}(\varepsilon x\, (A \wedge (B \wedge C)))$$
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Nested ε- and τ-terms as $\varepsilon x\, A(\varepsilon y\, B(x,y))$ express dependency among previous choices in their interpretation, allowing for embedding nested quantifier occurrences and represent all Skolem functions.

Epsilon terms are therefore strictly more expressive than f.o. quantifiers. Despite this, $\varepsilon\mathbf{P}$ is a conservative extension of both quantifier-free Predicate logic $\mathbf{P}$ and its quantified version $\mathbf{QP}$:

Theorem (1st Epsilon Theorem)

If $\vdash_{\varepsilon\mathbf{P}} A$ and A epsilon-* and quantifier-free**, then $\vdash_{\mathbf{P}} A$.

Theorem (2nd Epsilon Theorem)

If $\vdash_{\varepsilon\mathbf{P}} A$ and A epsilon-free*, then $\vdash_{\mathbf{QP}} A$.

*A formula is epsilon-free iff no ε- or τ-terms occur in it.

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Applications

Epsilon Calculus found many applications (Avigad and Zach, 2020):

- foundations of mathematics and mathematical logic
- philosophy of language and linguistics
- philosophy of science and of philosophy of mathematics
- automated theorem proving
- ...

In particular, philosophers of language and linguists thought of the indeterminate nature of epsilon terms as expressing indefinite descriptions of objects, in contrast with definite ones such as iota terms.

Using terms instead of connectives for quantification also allows for a simple, quantifier-free representation of anaphoric reference.

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- foundations of mathematics and mathematical logic
- philosophy of language and linguistics
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Hilbert's Program

In Hilbert's view, two components could be there distinguished in formalized mathematical theories:

- a 'real', finitistic part, characterized by statements expressing decidable properties;
- an 'ideal' part, characterized by statements representing unbounded quantifications that "have no meaning in themselves" (Hilbert, 1926).

The aim of Hilbert's Program was to prove the ideal part conservative over the real one, which is easily shown consistent.

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Logical Empiricism

In the logical empiricist tradition of philosophy of science, scientific theories were reconstructed distinguishing two components (Carnap, 1956):

- an 'empirical', observational part, characterized by terms ("hot", "blue") whose interpretation is known independently of the theory.
- a 'theoretical' part, characterized by terms ("temperature", "electric field") partially determined by theory laws only.

To provide an interpretation for theoretical components constituted the problem of theoretical terms.

A solution was finally given by Carnap (1961) providing explicit definitions for theoretical terms over epsilon terms:

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Schiemer and Gratzl (2016) and Andreas and Schiemer (2023) show how Carnap's method can be applied to mathematical structures:

Example (Adapted from Schiemer and Gratzl, 2016)

A monoid structure $\mathsf{M}(G, \circ, \mathsf{e})$ is axiomatized as follows:

- $\mathsf{e} \in G$
- $\forall x, y, z \in G$: $(x \circ y) \circ z = x \circ (y \circ z)$
- $\forall x \in G$: $x \circ \mathsf{e} = \mathsf{e} \circ x = x$

G, $\circ$ and e can be explicitly defined via epsilon terms:

$$G := \varepsilon X\, \exists f \exists u\, (u \in X \wedge \forall x, y, z \in X\colon (xfy)fz = xf(yfz) \wedge xfu = ufx = x)$$

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Applications of Epsilon Calculus rely on the f.o. formulation of epsilon terms, and hence do not scale to the propositional context.

Epsilon Calculus's underlying semantic machinery based on arbitrary choice functions can however be adapted in defining non-deterministically intensional contexts of evaluation witnessing a formula.

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Epsilon Modal logics extend a propositional language by modalities $\langle A \rangle$ and $[A]$, consisting of brackets $\langle , \rangle$ and $[,]$ resp., and an index formula A.

The intensional evaluation of formulas under the scope of ε- and τ-modalities relies on choice functions:

- for any Kripke model $\mathcal{M}$ based on frames $\mathcal{F} = \langle \mathcal{W}, \mathcal{R} \rangle$, consider all total choice functions ϕ s.t. $\phi(\emptyset)$ picks an arbitrary world in $\mathcal{W}$:

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Once again, world referents of ε- and τ-modalities remain indeterminate in evaluations:

Example

Let $\mathcal{M}$ be an EML model s.t. $\mathcal{W} = \{w, w_1, w_2\}$, $\mathcal{R} = wRw_1, wRw_2$, $P, Q \in w_1$ and $P, Q \in w_2$.

Then, $\mathcal{M}, w \vDash \langle P \rangle Q$, but the point of evaluation of Q is indeterminate:

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Axiomatization

When no frame conditions are imposed, the logic $\varepsilon\mathbf{K}$ is obtained by adding the following over a Propositional or a $\mathbf{K}$ Modal logic base:

WCrit	$\langle B\rangle A \to \langle A\rangle A$
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NEC	If $\vdash A$, then $\vdash [A]A$
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Theorem (Soundness and Completeness of $\varepsilon\mathbf{K}$)

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EML versions of well-known extensions of **K** are obtained adding their characteristic axioms. Remarkably, they are all conservative extensions over their Modal logics bases (proof later).

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Interestingly, the Epsilon Modal logic $\boldsymbol{\varepsilon}\mathbf{U}$ of universal frames (any two worlds relate) extends $\boldsymbol{\varepsilon}\mathbf{S5}$:[1]

U $[A]\neg B \leftrightarrow \neg[A]B$

This makes ε- and τ-modalities functional, i.e., distribute over any propositional connective, and simplifies Def:

$$\vdash_{\boldsymbol{\varepsilon}\mathbf{U}} \langle A\rangle(B \circ C) \leftrightarrow (\langle A\rangle B \circ \langle A\rangle C), \quad \textit{for } \circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$$

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The epsilon modalities in $\boldsymbol{\varepsilon}\mathbf{U}$ can be seen as indefinite descriptions of worlds labelling formulas, and hence expressible in Hybrid logic (Braüner, 2022) when extended by epsilon terms binding nominal variables.

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$$\mathcal{M}, w \vDash \langle\langle R\rangle Q\rangle P$$
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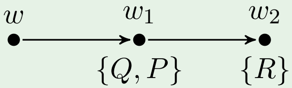

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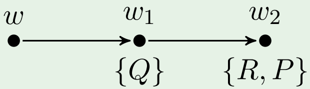

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Standard Translation

Any modal formula can be translated to a f.o. quantified Predicate one by the so-called standard translation ST (Blackburn et al., 2001):

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\begin{aligned}
\mathrm{ST}_x(P) &:= Px \\
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Theorem (Embedding of K in QP)

$\vDash_{\mathbf{K}} A$ *iff* $\vDash_{\mathbf{QP}} \forall x\,\mathrm{ST}_x(A)$

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The obtained εST is indexed by epsilon terms as well:

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Modal Translation

Fitting (2002) shows a modal embedding MT of **QP** over **λS5**, i.e., quantifier-free predicate **S5** with λ predicate abstraction and intension variables, denoted by i, whose interpretation is world-dependent:

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The formula index of an epsilon modality denotes a witness related world satisfying it. In this sense, epsilon modalities provide indefinite descriptions of points of evaluation.

From this, the previous embeddings show how applications of Epsilon Calculus can be reinterpreted in Epsilon Modal logics:

- by $\varepsilon\mathtt{ST}$, indefinite descriptions of points of evaluation are expressible as indefinite descriptions of objects;
- by $\varepsilon\mathtt{MT}$, indefinite descriptions of objects are expressible as interpretations at indefinitely described points of evaluation.

Moreover, being epsilon modalities connectives, these applications now generalize at the propositional level, and enjoy a clearer scoping mechanism than that of epsilon terms.

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The previous interpretations of epsilon terms as ideal objects in Hilbert's Program and explicit definitions of theoretical terms by Carnap can be generalized on intensional grounds as well.

Andreas and Schiemer (2016) show how Carnap's definitions can be given a modal semantics representing the set of theoretical terms possible referents as worlds. In mathematics, this reading supports an 'eliminative' Modal Structuralism interpretation (Andreas and Schiemer, 2023).

Epsilon Modal logics generalize Andreas and Schiemer's approach, but also support a straightforward structuralist interpretation, which seems in line with a so-called Concept Structuralism.

In this case, objects of mathematics are interpreted as 'ideal world' structures (Feferman, 2014) under a partial notion of truth.

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Epsilon Modal logics generalize Andreas and Schiemer's approach, but also support a straightforward structuralist interpretation, which seems in line with a so-called Concept Structuralism.

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Under this reading, epsilon modalities provide a logic for interpreting arbitray witness points of evaluation as denoting 'theoretical contexts' in which certain properties of structures hold:

Example

Consider again monoid structures $\mathsf{M}(G, \circ, \mathsf{e})$ axiomatized as:

- $\mathsf{e} \in G$
- $\forall x, y, z \in G$: $(x \circ y) \circ z = x \circ (y \circ z)$
- $\forall x \in G$: $x \circ \mathsf{e} = \mathsf{e} \circ x = x$

Commutative monoids add the following axiom:

- $\forall x, y \in G$: $x \circ y = y \circ x$

A commutative monoid structure can however be isolated as a monoid structure interpreted in a context supporting the commutativity of $\circ$:

$\langle \forall x, y \in G\colon x \circ y = y \circ x \rangle (\mathsf{e} \in G \land \forall x, y, z \in G\colon (x \circ y) \circ z = x \circ (y \circ z) \land x \circ \mathsf{e} = \mathsf{e} \circ x = x)$

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I reviewed Epsilon Calculus semantics based on arbitrary choice functions and its axiomatization, and showed some of its applications in the foundations of mathematics and philosophy.

Then, I showed how the semantics machinery underlying the interpretation of epsilon terms can be adapted in order to define intensional contexts of evaluation, and hence a new kind of epsilon modalities.

The resulting Epsilon Modal logics inherit many properties of Epsilon Calculus. Given the tight connection between the two systems, I showed how they can be embedded into each other.

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Finally, I showed how these embedding allow for applications of Epsilon Calculus to be generalized and applied at the propositional level.

In particular, the reading of epsilon terms in Hilbert's and Carnap's applications seem to conflate in a more general structuralist picture for mathematical objects that supports a notion of theoretical context. These can be expressed by epsilon modalities.

Given the straightforward correspondence between Epsilon Calculus and EMLs, these results generalize to other version of Epsilon Calculus, such as non-extensional ones.

These and Epsilon versions of other extension of Modal logics, such as non-normal and f.o. Predicate ones promise an even broader range of applications.

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