Hyperintensional Logics

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Hyperintensional reasoning

Frege's theory of meaning

Frege (1892) proposed a theory of meaning according to which, for each expression *e* of a given language, we should distinguish between:

- the object designated by *e*, if any (i.e., its denotation/*Bedeutung*);
- the mode in which *e* denotes an object (i.e., its sense/*Sinn*).

In the case of sentences, their denotation is a truth-value and their sense is a thought (which can be objectively assessed).

Hyperintensional reasoning

Direct contexts

We use both the expression "the morning star" and the expression "the evening star" to denote the planet Venus, and yet reference to such planet is made via different modes of presentation in the two cases.

In direct contexts, expressions having the same reference can be substituted *salva veritate*.

- Venus is the morning star;
- Venus is the evening star.

Hyperintensional reasoning

Indirect contexts

In indirect contexts, such as the scope of verbs for propositional attitudes, the substitution of expressions with the same reference is sometimes problematic.

- Eva knows that Venus is the morning star;
- Eva knows that Venus is the evening star.

Hyperintensional reasoning

Extensions and intensions

Carnap (1947) proposed an interpretation of modal logic according to which:

- the extension of a sentence can be regarded as its truth-value in a state of evaluation;
- the intension of a sentence is a function that assigns to it a truth-value in each state of evaluation.

Hyperintensional reasoning

Equi-intensionality
Two expressions have the same intension precisely when they have the same extension in every state of evaluation. In the case of sentences, equi-intensionality means *logical equivalence*.

There are contexts in which equi-intensional expressions cannot be substituted *salva veritate*. These are called *hyperintensional contexts* by Cresswell (1975).

Carnap (1947) already acknowledged the need for a finer-grained analysis of substitution in belief contexts via *intensional isomorphism*.

Hyperintensional reasoning

Hyperintensional contexts

The notion of hyperintensional context has been subsequently broadened, drawing inspiration from the following distinctions in relational semantics for modal logic:

1. necessary equivalence w.r.t. accessible states of evaluation;
2. valid equivalence w.r.t. a model;
3. valid equivalence w.r.t a class of models characterizing a system.

A flexible definition

Given a relation i (for $1 \leq i \leq 3$) and two sentences ϕ and ψ s.t. $i(\phi, \psi)$, an *i*-hyperintensional context is a context where the substitution of ψ for ϕ may fail to preserve truth.

$$3\text{-hyper.} \Rightarrow 2\text{-hyper.} \Rightarrow 1\text{-hyper.}$$

Examples

Mathematical ability

Mathematical truths are equi-intensional in a strong sense (w.r.t. every possible state of evaluation). Yet, one's ability to prove a mathematical theorem does not entail one's ability to prove all of them.

- Julia is able to prove that in Peano arithmetic the sum of a pair of odd numbers is an even number.
- Julia is able to prove that in Euclidean geometry the sum of angles of a triangle equals 180 degrees.

Examples

Content-stressing constructions

Certain natural language constructions are aimed at stressing specific content. Thus, expressions in their scope cannot be replaced by necessarily equivalent ones.

- it is important for tomorrow that you tidy up your room (t);
- it is important for tomorrow that you tidy up your room (t) and that Sting remains younger than Bob Dylan (y).

In our perspective, the possible (accessible) states for tomorrow where t is true are the same as those where $t \wedge y$ is true.

Examples

Explanations

Acknowledging that a sentence explains another does not allow one to substitute any of those sentences with necessarily equivalent ones.

- Mark was arrested because he had made a U-turn on the highway (u);
- Mark was arrested because he had made a U-turn on the highway (u) and he had not exceeded the speed of light (l).

The possible states where u is true are exactly the possible states where $u \wedge \neg l$ is true.

Examples

Obligations

In deontic logic there is a debate on whether the formal languages used represent explicit obligations or what holds in normatively ideal situations. In the former case, we have hyperintensional contexts:

- it is obligatory that all borrowed books are returned to the library by the end of the semester;
- it is obligatory that all borrowed books are returned to the library by the end of the semester and $2 + 2 = 4$.

Examples

Quotations

How far should we go in drawing distinctions? In the case of quotations, it is the syntax itself that blocks substitutions:

- the letter contains the text "a swift was seen flying over the hills while Nancy left her house";
- the letter contains the text "someone saw a swift flying over the hills while Nancy left Nancy's house".

Semantic notes

Metaphysics and formal semantics

Taking a stance on what hyperintensions *are* is a complex matter. To mention a few options (see Berto & Nolan 2021 and Sedlár 2021):

- modes of presentation;
- topics;
- structured contents;
- sets of truthmakers;
- procedures;
- cognitive relations between expressions and subjects;
- etc.

Here we just look at some approaches to their formal semantics.

Semantic notes

Levels of granularity

In order to account for all hyperintensional distinctions, one would probably need a logic framework stronger than the Predicate Calculus.

For instance, distinctions concerning active vs. passive voices of verbs, different positions occupied by operators in a sentence or anaphora would require a formal language able to keep track of those phenomena (see Ben-Yami 2014).

This is an open problem for future research. Here we restrict our attention to a simplified approach where the internal structure of sentences is not analysed.

Semantic notes

Semantics based on states of evaluation

There are various formal semantics for hyperintensional reasoning and they can be grouped in many ways (see the taxonomies in Sedlár 2021 and Wansing 1990). In the case of semantics based on *states of evaluation*, we can distinguish among:

- states that assign exactly one value in $\{1, 0\}$ to each atomic sentence (possible worlds);
- states that assign both values in $\{1, 0\}$ to some atomic sentence (impossible worlds);
- states that assign no value in $\{1, 0\}$ to some atomic sentence (incomplete worlds).

A semantics can rely on a combination of these options.

Semantic notes

Possible worlds and additional devices

Our focus is on semantics involving *possible worlds only*. These may make use of *additional devices*, such as:

- sets of distinguished sentences (e.g., those the agent is aware of; see Fagin & Halpern 1988);
- sets of relations among sentences and other parameters (e.g., expressing content pertinence in a context; see Glavaničová & Pascucci 2021)
- sets of topics (associated to formulas; see Hawke et al. 2020);
- algebraic structures (e.g., an abstract structure of thoughts; see Sedlár 2021);
- combinations of neighborhood functions and accessibility relations (see Chellas & Segerberg 1996).

Non-congruential modal logic

The rule of congruence

Hyperintensional logics can be defined also in a syntactic way. For instance, in modal logic failure of substitution for equi-intensional sentences corresponds to exceptions to the following rule of congruence:

(RC) If $\vdash \phi \leftrightarrow \psi$, then $\vdash \Box\phi \leftrightarrow \Box\psi$.

A system which is not closed under RC is said to be *non-congruential*.

Non-congruential modal logic

Semantics for non-congruential systems

Various semantics for non-congruential modal systems have been formulated over the years. Some are tailored to specific classes of systems (e.g., Chellas & Segerberg 1996 and Pietruszczak 2009); others aim at constituting a general framework (e.g., Rantala 1982 and Fagin & Halpern 1988).

Here we analyse the semantics formulated by Sedlár (2021) and further developed in Pascucci & Sedlár (2023). It employs *hyperintensional models* where states of evaluation are possible worlds.

Non-congruential modal logic

Formal setting

Let $\mathcal{P}$ be a propositional language of classical logic and $\mathrm{Mod}(\mathcal{P})$ a modal extension of it obtained with operator $\Box$.

For $\mathcal{X} \in \{\mathcal{P}, \mathrm{Mod}(\mathcal{P})\}$, an $\mathcal{X}$*-type algebra* is any algebra $\mathbf{A} = (Pr, \{c^{\mathbf{A}} \mid c \in Con_{\mathcal{X}}\})$. The wffs of $\mathcal{X}$ constitute an $\mathcal{X}$-type algebra $\mathbf{Fm}_{\mathcal{X}}$.

Given two algebras $\mathbf{A}$ and $\mathbf{B}$, an $\mathcal{X}$*-homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a mapping that preserves structure w.r.t. $Con_{\mathcal{X}}$.

Hyperintensional models

Definition (Hyperintensional model)

A *hyperintensional model* for $\mathrm{Mod}(\mathcal{P})$ is a tuple $\mathfrak{M} = (W, \mathbf{H}, O, N, I)$, where

- W is a non-empty set of states;
- $\mathbf{H}$ is a $\mathcal{P}$-type algebra of hyperintensions;
- O is a $\mathcal{P}$-homomorphism from $\mathbf{Fm}_{\mathrm{Mod}(\mathcal{P})}$ to $\mathbf{H}$, called hyperintension assignment;
- N is a function from $\mathbf{H}$ to $\wp(W)$, called necessity assignment;
- I is a $\mathcal{P}$-homomorphism from $\mathbf{H}$ to $\wp(W)$ called intension assignment and s.t. $I(O(\Box\varphi)) = N(O(\varphi))$

Hyperintensional models

Computing the semantic value

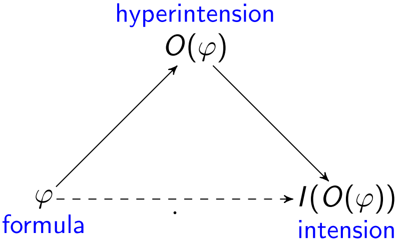

Figure: Sentence meaning in hyperintensional models.

Truth-conditions and validity

$\mathfrak{M}, w \vDash \phi$ iff $w \in I(O(\phi))$.

$\mathfrak{M} \vDash \phi$ iff $I(O(\phi)) = W$.

Formal results

General characterization strategy

Given a logic L based on a language $\mathcal{X}$, φ^L is the set of all maximal L-consistent theories Γ s.t. $\varphi \in \Gamma$.

Let $\mathfrak{M}^L = (W^L, \mathbf{H}^L, O^L, N^L, I^L)$ be s.t.:

- W^L is the set of all maximal L-consistent theories;
- $\mathbf{H}^L = \mathbf{Fm}_{\mathrm{Mod}(\mathcal{P})}$;
- $O^L(\varphi) = \varphi$;
- $N^L(\varphi) = (\Box\varphi)^L$;
- $I^L(O^L(\varphi)) = \varphi^L$.

$\mathfrak{M}^L$ is a (canonical) hyperintensional model.

Theorem (Characterization)

For each logic L over $\mathrm{Mod}(\mathcal{P})$ *and each ϕ from* $\mathbf{Fm}_{\mathrm{Mod}(\mathcal{P})}$, $\vdash_L \phi$ *iff* $\mathfrak{M}^L \vDash \phi$.

Formal results

Classes of hyperintensional models

We can identify classes of hyperintensional models in which the algebra of hyperintensions satisfies certain properties.

Definition (Boolean model)

A *Boolean model* is a hyperintensional model where **H** is a Boolean algebra.

Formal results

Additional properties

Properties of Boolean models can be added in a modular way to obtain a semantic characterization for specific non-congruential modal systems.

Take any x, y from $\mathbf{H}$ and let $x \leq^{\mathbf{H}} y$ mean $x \vee^{\mathbf{H}} y = y$. A Boolean model is said to be:

- monotonic iff $x \leq^{\mathbf{H}} y$ entails $N(x) \subseteq N(y)$;
- regular iff it is monotonic and $N(x) \cap N(y) \subseteq N(x \wedge^{\mathbf{H}} y)$;
- N-consistent iff $N(x) \cap N(\neg^{\mathbf{H}} x) = \emptyset$.

Formal results

Examples of non-congruential systems

The following systems contain the Propositional Calculus (PC) and are closed under Modus Ponens (for some, see Lemmon 1957):

- $B0$ is the weakest system that is closed under the rule (RC_{PC}) $\dfrac{\varphi \leftrightarrow \psi \in PC}{\Box\varphi \leftrightarrow \Box\psi}$
- $B1$ is the weakest system that is closed under the rule (RM_{PC}) $\dfrac{\varphi \to \psi \in PC}{\Box\varphi \to \Box\psi}$
- $C1$ is the weakest system including the axiom (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ that is closed under (RM_{PC})
- $D1$ is the weakest system including the axioms (K) and (D) $\Box\varphi \to \neg\Box\neg\varphi$ that is closed under (RM_{PC}).

Formal results

Interpretation via hyperintensional models

Theorem (Semantic characterization)

1. $\vdash_{B0} \varphi$ *iff* φ *is valid in all Boolean models.*
2. $\vdash_{B1} \varphi$ *iff* φ *is valid in all monotonic Boolean models.*
3. $\vdash_{C1} \varphi$ *iff* φ *is valid in all regular Boolean models.*
4. $\vdash_{D1} \varphi$ *iff* φ *is valid in all regular N-consistent Boolean models.*

Formal results

Final remarks

Adopting some terminology from Wansing (1990), the semantics at issue can be described as:

- *basic*, since (in its more general form) it can be used to semantically characterize the Propositional Calculus formulated over $\mathrm{Mod}(\mathcal{P})$;
- *general*, since properties can be added to classes of models in a modular way, thus characterizing modal systems with a different deductive power;
- *unifying*, since within this framework one can simulate related approaches developed in the literature.

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Thank you very much for your attention!